Mathematics for Computer Science
Spring 2019
Due: 23:59, March 25, 2019

# Homework Set 4

**Reading Assignments:** Read Chapter 6.8 – 6.10 of LPV.

**Written Assignments:** Do the following exercises from LPV: 6.10.22, 6.10.23.

**Special Problem 1** (counted as 1 exercise) Consider a *random walk* on a circle with nodes $v_0, v_1, \cdots, v_{n-1}$, and an edge between $v_i$ and $v_{(i+1) \bmod n}$ for each of $i = 0, 1, 2, \cdots, n-1$. Starting initially at $v_0$, in each step we move from the current node $v_i$ randomly to either $v_{(i-1) \bmod n}$ or $v_{(i+1) \bmod n}$ with equal probability. After $N$ steps, let $p_N$ be the probability that all $n$ nodes have been visited. Prove that $p_N \geq 1 - cn/N^{1/2}$ for some positive constant $c$.

**Remark** This implies that $p_N \to 1$ as $N \to \infty$. Thus, an infinite random walk on such a circle will eventually visit all the nodes.

**Special Problem 2** (counted as 4 exercises) Let $P_{2n}$ be the set of all $(2n)!$ permutations of $\{1, 2, 3, \cdots, 2n\}$. For any $\sigma = (a_1, a_2, \cdots, a_{2n}) \in P_{2n}$, a pair of positions $(i, j)$ (where $i < j$) is called an *inversion* in $\sigma$ if $a_i > a_j$. For example, in the permutation $(a_1, a_2, a_3, a_4) = (2, 4, 1, 3)$, $(2, 4)$ is an inversion as $a_2 = 4 > a_4 = 3$; in fact, in this case there are exactly 3 inversions $(1, 3), (2, 3), (2, 4)$.

For any $\sigma = (a_1, a_2, \cdots, a_{2n}) \in P_{2n}$, let $f(\sigma)$ be the permutation obtained from $\sigma$ by sorting the sublist of odd positions. That is, $f(\sigma) = (b_1, b_2, \cdots, b_{2n}) \in P_{2n}$, where $b_k = a_k$ for $k = 2, 4, 6, \cdots, 2n$ and $b_1 < b_3 < b_5 < \cdots < b_{2n-1}$ is the sorted list of $a_1, a_3, \cdots, a_{2n-1}$. For example, for $\sigma = (3, 8, 2, 5, 6, 7, 1, 4)$, $f(\sigma) = (1, 8, 2, 5, 3, 7, 6, 4)$. For a random $\sigma$ uniformly chosen from $P_{2n}$, let $I_n$ be the random variable corresponding to the number of inversions in the permutation $f(\sigma)$. Do the following problems:
(a) (counted as 1 exercise) Determine $E(I_n)$.
(b) (counted as 3 exercises) Determine $Var(I_n)$.

*Remarks* For Question (a) above, let $h_n = E(I_n)$ be the answer, then $h_1 = 1/2$, $h_2 = 13/6$, $h_3 = 5$. Your formula should be consistent with that.

**Special Problem 3** (counted as 4 exercises) Let $X_1, X_2, \cdots, X_n$ be independent Poisson trials such that $\Pr\{X_i = 1\} = p_i$. Let $X = \sum_{1 \leq i \leq n} X_i$ and $\mu = E(X)$. In class we derived one version of the Chernoff Bounds regarding the probability that $X > (1 + \delta)\mu$. Here you are asked to prove the following bounds in a similar way:

(a) For $0 < \delta < 1$,

$$\Pr\{X \leq (1-\delta)\mu\} \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

(b) Assume that $p_i = 1/2$ for all $i$. Prove the stronger bound that

$$\Pr\{|X - \frac{n}{2}| > a\} \leq 2e^{\frac{-2a^2}{n}}.$$

(**Hint:** First show that $e^t + 1 \leq 2e^{t/2+t^2/8}$ for all $t > 0$.)

**Special Problem 4** (counted as 2 exercises) Use the Chernoff Bounds derived in class and in the above problem to prove the following inequalities: For all $0 < \delta \leq 1$
(a) $\Pr\{X \geq (1+\delta)\mu\} \leq e^{-\mu\delta^2/3}$.
(b) $\Pr\{X \leq (1-\delta)\mu\} \leq e^{-\mu\delta^2/2}$.
**Remark** Note that it follows from (a) and (b) that $\Pr\{|X - E(X)| > a\} \leq 2e^{-a^2/3E(X)}$ for all $0 < a \leq E(X)$.