Mathesmatics for Computer Science: Homework 4

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LPV 6.10.22

We are given $n+1$ numbers from the set $\{1, 2, \cdots, 2n\}$. Prove that there are two numbers among them such that one divides the other.

Answer:

Divide each of the numbers a_i into groups A_k by $a_i = 2^p \cdot k$ with bigest integer p, which is:

$$A_k = 2^0 k + 2^1 k + 2^2 k + \cdots$$

Obviously every k is $k \equiv 1(mod 2)$, and we have a n-segmentation of $\{1, 2, \cdots, 2n\}$:

$$\begin{aligned} A_1 &= 2^0 \cdot 1, 2^1 \cdot 1, 2^2 \cdot 1, \cdots; \\ A_3 &= 2^0 \cdot 3, 2^1 \cdot 3, 2^2 \cdot 3, \cdots; \\ &\cdots \\ A_{2n-1} &= 2^0 \cdot (2n-1); \end{aligned}$$

And every two numbers in each group will have one can be divided by the other. So if we are given $n+1$ numbers, at least 2 are from one group, with pigeon hole principle known. So there are two numbers among them such that one divides the other.

LPV 6.10.23

What is the number of positive integers not larger than 210 and not divisible by 2, 3 or 7?

Answer: Similar to **6.9.1**, we have

$$210 - (\frac{210}{2} + \frac{210}{3} + \frac{210}{7}) + (\frac{210}{2 \cdot 3} + \frac{210}{2 \cdot 7} + \frac{210}{3 \cdot 7}) - \frac{210}{2 \cdot 3 \cdot 7} = 60$$

integers not larger than 210 and not divisible by 2, 3 or 7.

Special Problem 3

Let $X_1, X_2, \cdots, X_n$ be independent Poisson trials such that $Pr\{X_i = 1\} = p_i$. Let $X = \sum_{1 \le i \le n} X_i$ and $\mu = E(X)$. In class we derived one version of the Chernoff Bounds regarding the probability that $X > (1 + \sigma)\mu$. Here you are asked to prove the following bounds in a similar way:

(a) For $0 < \delta < 1$,

$$Pr\{X \leq (1-\delta)\mu\} \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

(b) Assume that $p_i = 1/2$ for all i. Prove the stronger bound that

$$Pr|X - \frac{n}{2}| > a \leq 2e^{\frac{-2a^2}{n}}.$$

(Hint: First show that $e^t + 1 \leq 2e^{t/2+t^2/8}$ for all $t > 0$.)

Answer:

(a) Known $Pr(X_i = 1) = p_i$ and $Pr(X \leq a) \leq e^{ta}\Pi_i E[e^{-tX_i}]$ with $t > 0$.

$$\begin{aligned} Pr(X \leq (1-\delta)\mu) &\leq \frac{\Pi_{i=1}^n E[e^{-tX_i}]}{e^{-t(1-\delta)\mu}} \\ &= \frac{\Pi_{i=1}^n [p_i e^{-t} + (1-p_i)]}{e^{-t(1-\delta)\mu}} \end{aligned}$$

And with know $1 + x \leq e^x$, we have $p_i e^{-t} + (1-p_i) = p_i(e^{-t} - 1) + 1 \leq e^{p^i(e^{-t}-1)}$. So

$$\begin{aligned} Pr(X \leq (1-\delta)\mu) &\leq \frac{\Pi_{i=1}^n e^{p_i(e^{-t}-1)}}{e^{-t(1-\delta)\mu}} \\ &= \frac{e^{(e^{-t}-1)\sum_{i=1}^n p_i}}{e^{-t(1-\delta)\mu}} \\ &= \frac{e^{(e^{-t}-1)\mu}}{e^{-t(1-\delta)\mu}} \end{aligned}$$

Set $t = -\ln(1-\delta)$ and $t > 0$ when $0 < \delta < 1$, then

$$\begin{aligned} Pr(X \leq (1-\delta)\mu) &\leq \frac{e^{(e^{-t}-1)\mu}}{e^{-t(1-\delta)\mu}} \\ &= \frac{e^{-\delta\mu}}{(1-\delta)^{(1-\delta)\mu}} \\ &= \left[\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right]^{\mu} \end{aligned}$$

(b) I have no idea from $e^t - 1 \leq 2e^{t/2+t^2/8}$. But a different prove from **Probability and Computing: Randomized Algorithms and Probabilistic Analysis** is found as below:

Let $Y_1, Y_2, \cdots, Y_n$ be independent random variables with $Pr(Y_i = 1) = Pr(Y_i = -1) = \frac{1}{2}$ and $Y = \sum_{i=1}^n Y_i$, for any $t > 0$,

$$\begin{aligned} E[e^{tY_i}] &= \frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ &= \frac{1}{2}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots) - \frac{1}{2}(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots) \\ &= \sum_{i \geq 0} \frac{t^{2i}}{(2i)!} \\ &= \sum_{i \geq 0} (t^2/2)^i / i! \\ &= e^{t^2/2}. \end{aligned}$$

With $t = \frac{a}{n} > 0$, we get

$$\begin{aligned} Pr(Y \geq a) &\leq \frac{E[e^{tY}]}{e^{ta}} \\ &= \frac{\Pi_{i=1}^{n} E[e^{tY_i}]}{e^{ta}} \\ &= e^{nt^2/2 - ta} \\ &= e^{-a^2/2n} \end{aligned}$$

So with X_i have $Pr(X_i = 1) = Pr(X_i = 0) = \frac{1}{2}$ and $X = \sum_{i=1}^{n} X_i$. We get $\mu = \frac{n}{2}$ and $X = \frac{1}{2}\sum_{i=1}^{n}(X_i + 1) = \frac{1}{2}Y + \mu$

$$Pr(X \geq \mu + a) = Pr(Y \geq 2a) \leq e^{-4a^2/2n}$$

with symmetry, we finally get

$$Pr(|X - \frac{n}{2}| \geq a) \leq 2e^{\frac{-2a^2}{n}}$$

Special Problem 4

Use the Chernoff Bounds derived in class and in the above problem to prove the following inequalities: For all $0 < \delta \leq 1$

(a) $Pr\{X \geq (1+\delta)\mu\} \leq e^{-\mu\delta^2/3}$.

(b) $Pr\{X \leq (1-\delta)\mu\} \leq e^{-\mu\delta^2/2}$

Remark Note that it follows from (a) and (b) that $Pr\{|X - E(X)| > a\} \leq 2e^{-a^2/3E(X)}$ for all $0 < a \leq E(X)$.

Answer:

(a) From SP3a we get a symmetry formula:

$$Pr(X \geq (1+\delta)\mu) \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu}$$

To get

$$Pr(X \geq (1+\delta)\mu) \leq e^{-\mu\delta^2/3}$$

We can get

$$\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \leq e^{-\delta^2/3}$$

first.

The derivative of upper inequality is writen as blow:

$$f(\delta) = \delta - (1+\delta)\ln(1+\delta) + \frac{\delta^2}{3} \leq 0$$

So

$$\begin{aligned} f'(\delta) &= -\ln(1+\delta) + \frac{2}{3}\delta \\ f''(\delta) &= -\frac{1}{1+\delta} + \frac{2}{3} \end{aligned}$$

So $f''(\delta) < 0$ for $0 < \delta < \frac{1}{2}$ and $f''(\delta) > 0$ for $\frac{1}{2} < \delta \leq 1$. And $f'(0) = 0$, $f'(1) = -\ln(2) + \frac{2}{3} < 0$, so $f'(\delta) < 0$ for all $0 < \delta \leq 1$.

With $f(0) = 0$, we are convienced now that $f(\delta) < 0$ for all $0 < \delta \leq 1$, which equals to (a).

(b) Similary to (a), we get the derivative as below:

$$g(\delta) = -\delta - (1-\delta)\ln(1-\delta) + \frac{\delta^2}{2} \leq 0$$

And

$$g'(\delta) = \ln(1-\delta) + \delta$$
$$g''(\delta) = 1 - \frac{1}{1-\delta}$$

Obviously, $g''(\delta) < 0$ for all $0 < \delta \leq 1$. And $g'(0) = 0$, so $g'(\delta) < 0$; $g(0) = 0$, so $g(\delta) < 0$ is got easily.

Acknowledgement:

SP3a: Wikipedia Chernoff bound

SP3b: Part 4.8 and 4.9 from book Probability and Computing: Randomized Algorithms and Probabilistic Analysis