Mathematics for Computer Science
Spring 2019
Due: 23:59, April 1, 2019

Homework Set 5

Do the following special problems:

Special Problem 1 (counted as 3 exercises) [*Inference in Bayesian Networks*]

Consider a probability space with three binary (i.e. $0-1$ valued) random variables B, F, G satisfying the following properties:
(1) B and F are independent with $\Pr\{B = 1\} = \Pr\{F = 1\} = 0.9$;
(2) $\Pr\{G = 1 \,|\, B = 1, F = 1\} = 0.8$, $\Pr\{G = 1 \,|\, B = 1, F = 0\} = 0.2$, $\Pr\{G = 1 \,|\, B = 0, F = 1\} = 0.2$, $\Pr\{G = 1 \,|\, B = 0, F = 0\} = 0.1$.

Remarks The above random variables model a car system with B, F representing the true physical states of the battery and fuel respectively (value 1 representing *full*, while 0 representing *empty*). The random variable G represent a fuel gauge (i.e. a meter on the panel that the car driver can see), which gives a (somewhat unreliable) reading of the fuel state. Both B and F can influence (probabilistically) the result of reading the gauge G. Thus, the three random variables (together with the probability assignments) form a very simple *Bayesian network*, with a causal arrow pointing from B to G, and an arrow pointing from F to G.

Questions:
(a) (1 exercise) Even without calculation, we expect the value of $\Pr\{F = 0 \,|\, G = 0\}$ to be greater than $\Pr\{F = 0\}$. Why? Now for the calculation. What is $\Pr\{G = 0\}$? What is $\Pr\{F = 0\}$? What is $\Pr\{G = 0 \,|\, F = 0\}$? Use Bayes' Rule to determine the value of $\Pr\{F = 0 \,|\, G = 0\}$.
(b) (1 exercise) Even without calculation, we expect the value of $\Pr\{F = 0 \,|\, G = 0, B = 0\}$ to be less than $\Pr\{F = 0 \,|\, G = 0\}$ (which was calculated in (a) above). Why? Determine exactly the value of $\Pr\{F = 0 \,|\, G = 0, B = 0\}$.
(c) (1 exercise) Give an explicit description of a probability space (Ω, p) and the realization of the random variables B, F, G in that probability space.

Special Problem 2 (counted as 5 exercises) [*Randomized Routing*]

In class, we discuss a routing algorithm on the $n - bit$ hypercube, called *bit fixing algorithm*, for a node i to send a message to a node k using $d(i, k)$ edges where $d(i, k)$ is the Hamming distance between i and k. Let σ be a permutation so that for each node $i \in \{0, 1\}^n$ in the hypercube network, a message packet m_i is to be routed to node $\sigma(i)$ (starting in parallel at the same time, as described in class). For each node j, let $\rho_j = e_1 e_2 \cdots e_{\ell_j}$ be the path (i.e. the sequence of edges) followed by packet m_j under the bit-fixing algorithm. Now let i be any fixed node. Let S be the set of $j \neq i$ such that the paths ρ_j and ρ_i share at least one common edge. The following theorem is important for the analysis of the randomized routing algorithm described in the last class.

Theorem A The number of steps used in delivering packet m_i is no more than $\ell_i + |S|$. (That is, the extra delay for packet ν_i is at most $|S|$.)

Questions:

(a) (1 exercise) Prove Theorem A for the special case $|S| = 1$.
(b) (1 exercise) Prove Theorem A for the special case $|S| = 2$.
(c) (3 exercises) Prove Theorem A for any $|S|$.