Mathematics for Computer Science: Homework 5

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Special Problem 1

[*Inference in Bayesian Networks*]

Consider a probability space with three binary (i.e. 0-1 valued) random variables B, F, G satisfying the following properties:

(1) B and F are independent with $Pr\{B = 1\} = Pr\{F = 1\} = 0.9$;

(2) $Pr\{G = 1 \mid B = 1, F = 1\} = 0.8$, $Pr\{G = 1 \mid B = 1, F = 0\} = 0.2$, $Pr\{G = 1 \mid B = 0, F = 1\} = 0.2$, $Pr\{G = 1 \mid B = 0, F = 0\} = 0.1$.

Remarks The above random variables model a car system with B, F representing the true physical states of the battery and fuel respectively (value 1 representing full, while 0 representing empty). The random variable G represent a fuel gauge (i.e. a meter on the panel that the car driver can see), which gives a (somewhat unreliable) reading of the fuel state. Both B and F can influence (probabilistically) the result of reading the gauge G. Thus, the three random variables (together with the probability assignments) form a very simple Bayesian network, with a causal arrow pointing from B to G, and an arrow pointing from F to G.

Questions:

(a) Even without calculation, we expect the value of $Pr\{F = 0 \mid G = 0\}$ to be greater than $Pr\{F = 0\}$. Why? Now for the calculation. What is $Pr\{G = 0\}$? What is $Pr\{F = 0\}$? What is $Pr\{G = 0 \mid F = 0\}$? Use Bayes' Rule to determine the value of $Pr\{F = 0 \mid G = 0\}$.

(b) Even without calculation, we expect the value of $Pr\{F = 0 \mid G = 0, B = 0\}$ to be less than $Pr\{F = 0 \mid G = 0\}$ (which was calculated in (a) above). Why? Determine exactly the value of $Pr\{F = 0 \mid G = 0, B = 0\}$.

(c) Give an explicit description of a probability space (Ω, p) and the realization of the random variables B, F, G in that probability space.

Answer:

We know that:

$$\begin{aligned}
Pr\{B = 1\} = Pr\{F = 1\} = 0.9,\ & Pr\{B = 0\} = Pr\{F = 0\} = 0.1 \\
Pr\{G = 1 \mid B = 1, F = 1\} = 0.8,\ & Pr\{G = 0 \mid B = 1, F = 1\} = 0.1 \\
Pr\{G = 1 \mid B = 1, F = 0\} = 0.2,\ & Pr\{G = 0 \mid B = 1, F = 0\} = 0.8 \\
Pr\{G = 1 \mid B = 0, F = 1\} = 0.2,\ & Pr\{G = 0 \mid B = 0, F = 1\} = 0.8 \\
Pr\{G = 1 \mid B = 0, F = 0\} = 0.1,\ & Pr\{G = 0 \mid B = 0, F = 0\} = 0.9
\end{aligned}$$

(a) Because from Bayes' Theroem, we have $Pr\{F = 0 \mid G = 0\}$ that:

$$Pr\{F = 0 \mid G = 0\} = \frac{Pr\{G = 0 \mid F = 0\}Pr\{F = 0\}}{Pr\{G = 0\}} = \frac{Pr\{G = 0 \mid F = 0\}}{Pr\{G = 0\}}Pr\{F = 0\}$$

And with known that G represent a fuel gauge, so $Pr\{G = 0 \mid F = 0\}$ should be greater than $Pr\{G = 0\}$, so $Pr\{F = 0 \mid G = 0\}$ to be greater than $Pr\{F = 0\}$.

$Pr\{G = 0\}$ can be get by:

$$\begin{aligned} Pr\{G = 0\} =& Pr\{G = 0 \mid B = 1, F = 1\}Pr\{B = 1\}Pr\{F = 1\} \\ &+ Pr\{G = 0 \mid B = 1, F = 0\}Pr\{B = 1\}Pr\{F = 0\} \\ &+ Pr\{G = 0 \mid B = 0, F = 1\}Pr\{B = 0\}Pr\{F = 1\} \\ &+ Pr\{G = 0 \mid B = 0, F = 0\}Pr\{B = 0\}Pr\{F = 0\} \\ =& 0.2 \times 0.9 \times 0.9 + 0.8 \times 0.9 \times 0.1 + 0.8 \times 0.1 \times 0.9 + 0.9 \times 0.1 \times 0.1 \\ =& 0.315 \end{aligned}$$

And obviously, we have:

$$Pr\{F = 0\} = 1 - Pr\{F = 1\} = 0.1$$

$$\begin{aligned} Pr\{G = 0 \mid F = 0\} =& Pr\{G = 0 \mid B = 1, F = 0\}Pr\{B = 1\} + Pr\{G = 0 \mid B = 0, F = 0\}Pr\{B = 0\} \\ =& 0.8 \times 0.9 + 0.9 \times 0.1 \\ =& 0.81 \end{aligned}$$

Finally,

$$\begin{aligned} Pr\{F = 0 \mid G = 0\} =& \frac{Pr\{G = 0 \mid F = 0\}Pr\{F = 0\}}{Pr\{G = 0\}} \\ =& \frac{0.81 \times 0.1}{0.315} \\ =& 0.257 \end{aligned}$$

(b) Because $G = 0$ has two sources: $B = 0$ and $F = 0$. So when $G = 0$, $F = 0$ has less probability when $B = 0$. To calculate $Pr\{F = 0 \mid G = 0, B = 0\}$, first we get $Pr\{G = 0 \mid B = 0\}$ by:

$$\begin{aligned} Pr\{G = 0 \mid B = 0\} =& Pr\{G = 0 \mid B = 0, F = 1\}Pr\{F = 1\} + Pr\{G = 0 \mid B = 0, F = 0\}Pr\{F = 0\} \\ =& 0.8 \times 0.9 + 0.9 \times 0.1 \\ =& 0.81 \end{aligned}$$

Then,

$$\begin{aligned} Pr\{F = 0 \mid G = 0, B = 0\} =& \frac{Pr\{G = 0, B = 0 \mid F = 0\}Pr\{F = 0\}}{Pr\{G = 0, B = 0\}} \\ =& \frac{Pr\{G = 0 \mid B = 0, F = 0\}Pr\{B = 0\}Pr\{F = 0\}}{Pr\{G = 0 \mid B = 0\}Pr\{B = 0\}} \\ =& \frac{0.9 \times 0.1}{0.81} \\ =& 0.111 < 0.257 = Pr\{F = 0 \mid G = 0\} \end{aligned}$$

(c) To descripe the probability space $(\Omega, \mathcal{F}, P)$, all $\mathcal{F}$ and their probabilities should be given:

$$\begin{aligned} &Pr\{B = 1\} = 0.9,\ Pr\{F = 1\} = 0.9, Pr\{G = 1\} = 0.685 \\ &Pr\{B = 0\} = 0.1,\ Pr\{F = 0\} = 0.1, Pr\{G = 0\} = 0.315 \end{aligned}$$

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$$Pr\{B=1,F=1\}=0.81, Pr\{B=1,F=0\}=0.09$$
$$Pr\{B=0,F=1\}=0.09, Pr\{B=0,F=0\}=0.01$$

$$Pr\{G=1,F=1\}=0.666, Pr\{G=1,F=0\}=0.019$$
$$Pr\{G=0,F=1\}=0.234, Pr\{G=0,F=0\}=0.081$$

$$Pr\{G=1,B=1\}=0.666, Pr\{G=1,B=0\}=0.019$$
$$Pr\{G=0,B=1\}=0.234, Pr\{G=0,B=0\}=0.081$$

$$Pr\{G=1,B=1,F=1\}=0.648, Pr\{G=1,B=1,F=0\}=0.018$$
$$Pr\{G=1,B=0,F=1\}=0.018, Pr\{G=1,B=0,F=0\}=0.001$$
$$Pr\{G=0,B=1,F=1\}=0.162, Pr\{G=0,B=1,F=0\}=0.072$$
$$Pr\{G=0,B=0,F=1\}=0.072, Pr\{G=0,B=0,F=0\}=0.009$$

Special Problem 2

[*Randomized Routing*]

In class, we discuss a routing algorithm on the $n-$bit hypercube, called bit fixing algorithm, for a node i to send a message to a node k using $d(i,k)$ edges where $d(i,k)$ is the Hamming distance between i and k. Let σ be a permutation so that for each node $i \in \{0,1\}$ n in the hypercube network, a message packet m_i is to be routed to node $\sigma(i)$ (starting in parallel at the same time, as described in class). For each node j, let $\rho_j = e_1 e_2 \cdots e_{l_j}$ be the path (i.e. the sequence of edges) followed by packet m_j under the bit-fixing algorithm. Now let i be any fixed node. Let S be the set of $j \neq i$ such that the paths ρ_j and ρ_i share at least one common edge. The following theorem is important for the analysis of the randomized routing algorithm described in the last class.

Theorem A The number of steps used in delivering packet m_i is no more than $l_i + |S|$. (That is, the extra delay for packet ν_i is at most $|S|$.)

Questions:

(a) Prove Theorem A for the special case $|S| = 1$.

(b) Prove Theorem A for the special case $|S| = 2$.

(c) Prove Theorem A for any $|S|$.

Answer:

First, we can know for sure that, if ρ_i and ρ_j meet and then diverge, they will not meet agin.

(a) Fix $i \in \{0,1\}^n$ and $\rho_i = (e_1, e_2, \cdots, e_{l_i})$. When $|S| = 1$, there is one node j that ρ_j shares part of edges for example $(e_{b_j}, \cdots, e_{d_j})$ with ρ_i. So there will be no congestion through all edges except $(e_{b_j}, \cdots, e_{d_j})$ from i to $\sigma(i)$. And when congestion occurs at $(e_{b_j}, \cdots, e_{d_j})$, there will be no more than 1 step to wait because only one competitive package m_j from node j will go through it. Thus, here we get $delay(i) \leq |S|$ for $|S| = 1$.

(b) When $|S| = 2$, there is two node j and k that ρ_j shares $(e_{b_j}, \cdots, e_{d_j})$ and ρ_k shares $(e_{b_k}, \cdots, e_{d_k})$ with $\rho_i = (e_1, e_2, \cdots, e_{l_i})$. Obviously, there will be no congestion through all edges except the union of the two parts of edges. And if congestion occurs at meeting edges of ρ_i and ρ_j or ρ_i and ρ_k, there will be at most 1 step delay from each meets. Because there will be only one package from each competitive node need to go through their meet edge. Thus, we get $delay(i) \leq |S|$ for $|S| = 2$ proved.

(c) First, a quantity shows the delay at each time step is defined as below:(i is waiting at the head of e_k when time step t begins.)

$$delay_t(i) = t - k$$

If $delay_t(i) = d$ and $delay_{t+1}(i) = d + 1 > 0$, let us say that one package m_j from a node j is ejected with at d.

Let T be the time step in which i reaches its destination. Note that $delay_1(i) = 0$ and if i reaches its destination in T time steps, then

$$delay_T(i) = T - l_i = \text{delay of } i.$$

We need to prove that $delay_T(i) \leq |S|$. If $delay_T(i) = D$, there must be nodes ejected at $d = 1, 2, \cdots, D$, meaning that $D \leq |S| =$ all nodes can be ejected. Here we get $delay(i) \leq |S|$ proved for any $|S|$.

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No thanks