Mathematics for Computer Science: Homework 6

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Special Problem 1

Show that for some fixed constants c, $c' > 0$, the randomized routing algorithm discussed in class has the following performance for Phase 2:

$$Pr\{T > cn\} \le 2^{-c'n}$$

Answer:

From **Homework 4: SP3a**, we have Chernoff bound as below:

$$Pr\{X \ge \beta\mu\} \le \left(\frac{e^{\beta-1}}{\beta^{\beta}}\right)^{\mu}$$

And in randomized routing algorithm, we have $E[T] \le 2n$, so:

$$Pr\{T > cn\} \le \left(\frac{e^{c/2-1}}{(c/2)^{c/2}}\right)^{2n} = \left(\frac{c}{2e}\right)^{-cn} e^{-2n}$$

So if $c \ge 4e$, there is a c' satisfies $Pr\{T > cn\} \le 2^{-c'n}$.

Special Problem 2

Solve each of the following recurrence relations:

(a) $a_0 = 1,\ a_1 = 2,\ a_n = 4a_{n-1} - 3a_{n-2} + 3n + 1$ for all $n \ge 2$.

(b) $a_0 = 1,\ a_n = \frac{a_{n-1}}{1+3a_{n-1}}$ for $n \ge 1$.

Answer:

(a) First, try to get the recursive equation into a simpler form that $b_n = 4b_{n-1} - 3bn - 2$. It's easy to assume $b_n = a_n + bn^2 + cn + d$. So:

$$(a_n + bn^2 + c_n + d) = 4(a_{n-1} + b(n-1)^2 + c(n-1) + d) - 3(a_{n-2} + b(n-2)^2 + c(n-2) + d)$$

Which get:

$$a_n = 4a_{n-1} - 3a_{n-2} + 4bn - 8b + 2c$$

Compare the coefficients, we get:

$$4b = 3 \qquad b = \frac{3}{4}$$
$$-8b + 2c = 1 \qquad c = \frac{7}{2}$$

Assume $b_n = a_n + \frac{3}{4}n^2 + \frac{7}{2}n$, and $b_0 = a_0 = 1$, $b_1 = a_1 + \frac{3}{4} + \frac{7}{2} = \frac{25}{8}$. And there is $b_n = 4b_{n-1} - 3b_{n-2}$. Let $b_n = z^n$, we get

$$z^2 = 4z - 3$$

The roots are: $z_1 = 1,\ z_2 = 3$, so $b_n = A + B3^n$. Getting with $b_0 = 1$, $b_1 = \frac{25}{8}$, we have $A = -\frac{13}{8}$, $B = \frac{21}{8}$. So finnaly, we get a_n as:

$$a_n = \frac{21}{8}3^n - \frac{3}{4}n^2 - \frac{7}{2}n - \frac{13}{8}$$

(b) Consider the characteristic equation:

$$\lambda = \frac{\lambda}{1 + 3\lambda}$$

Get roots $\lambda_1 = \lambda_2 = 0$. And we have a_n is never zero for the reason that $a_0 = 1 \neq 0$. So:

$$\frac{1}{a_n} = \frac{1 + 3a_{n-1}}{a_{n-1}} = \frac{1}{a_{n-1}} + 3$$

So obviously $\frac{1}{a_n} = 3n + 1$ so $a_n = \frac{1}{3n+1}$.

Special Problem 3

Consider a sequence of $2n$ people in a line at a cashier. Suppose n of the people pay 1 yuan each and n of the people get 1 yuan each. A *paying pattern* is a binary sequence $\sigma = a_1a_2\cdots a_{2n}$ with exactly n 1's and n 0's; the interpretation is that $a_j = 1$ if person j pays 1 yuan, and $a_j = 0$ otherwise. Note that there are exactly $\binom{2n}{n}$ paying patterns. Let b_n denote the number of paying patterns in which the cashier never goes in debt (i.e., at every stage at least as many people have paid in 1 yuan as were paid out 1 yuan).

(a) Derive a recurrence equation for b_n , and find an explicit expression for the generating function $\sum_{n\geq 1} b_n x^n$. Determine a closed-form solution for b_n.

(b) Use an alternative way (a *combinatorial proof*) to establish a closed form solution $b_n = \binom{2n}{n} - \binom{2n}{n+1}$. (**Hint:** Show a one-to-one correspondence between paying patterns where at some stage the cashier goes at least 1 yuan in debt and all binary sequences of length $2n$ with exactly $n+1$ 1's.)

Answer:

(a) Counting by hands, it's easily to get $b_1 = 1$, $b_2 = 2$, $b_3 = 5$.

Consider the sequence $\sigma = a_1a_2\cdots a_{2n}$ in which the cashier never goes in debt, then $a_1 = 1$ and since there are n 1's and n 0's, there must be a $1 \leq j \leq 2n-1$ that $[a_j, a_{j+1}] = [1, 0]$.

Treat the money in and out as a **stack**. Then *the first yuan* the cashier got can be paid out only at an even turn, denote as *the first yuan* is the l_{th} got paid out at $2l$ turn. Then there are still $l-1$ yuan paid before *the first yuan* and $n-l$ yuan to be paid after *the first yuan*. Thus, the sequence is divided into two unkown parts $a_2, \cdots, a_{2l-1}$ and $a_{2l+1}, a_{2l+2}, \cdots, a_{2n}$, and both of them satisfies the original rule.

So the recurrence is: (set $b_0 = 1$ for comprensively understanding)

$$b_{n+1} = \sum_{i=0}^{n} b_i b_{n-i}$$

Special Problem 3 continued on next page. . .

The generating function is:

$$\begin{aligned} f(x) &= \sum_{i\geq 0} b_n x^n = b_0 + b_1 x + b_2 x^2 + b3x^3 + \cdots \\ f^2(x) &= \left(\sum_{i\geq 0} b_n x^n\right)\left(\sum_{i\geq 0} b_n x^n\right) \\ &= b_0b_0 + (b_0b_1 + b_1b_0)x + (b_0b_2 + b_1b_1 + b_2b_0)x^2 + \cdots \\ &= \sum_{n\geq 1} b_n x^{n-1} \end{aligned}$$

So $f(x) - xf^2(x) = b_0 = 1$, get the roots:

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Known that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1 \pm \sqrt{1-4x}}{2x} = b_0 = 1$$

We get $f(x) = \frac{1-\sqrt{1-4x}}{2x}$, so $\sum_{n\geq 1} b_n x^n = f(x) - b_0 = \frac{1-\sqrt{1-4x}}{2x} - 1$. With $(1+x)^\alpha = 1 + \sum_{n=1}^{\infty}\binom{n}{\alpha}x^n$, we have:

$$\begin{aligned} (1-4x)^{\frac{1}{2}} &= 1 + \sum_{n=1}^{\infty}\binom{n}{1/2}(-4x)^n \\ &= 1 + \sum_{n=1}^{\infty}\frac{\frac{1}{2}\cdot(\frac{1}{2}-1)\cdots(\frac{3}{2}-n)}{n!}(-4x)^n \\ &= 1 - \sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-3)}{n!}2^n x^n \\ &= 1 - \sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-3)}{n!}\frac{2\cdot 4\cdots(2n-2)}{(n-1)!}2x^n \\ &= 1 - 2\sum_{n=1}^{\infty}\frac{(2n-2)!}{n!(n-1)!}x^n \end{aligned}$$

So:

$$\begin{aligned} \frac{1-\sqrt{1-4x}}{2x} &= \sum_{n=1}^{\infty}\frac{(2n-2)!}{n!(n-1)!}x^{n-1} \\ &= \sum_{n=0}^{\infty}\frac{(2n)!}{(n+1)!n!}x^n \end{aligned}$$

Finally we get: $b_n = \frac{(2n)!}{(n+1)!n!}x^n$.

(b) Consider the sequence $\sigma = a_1a_2\cdots a_{2n}$ and call it wanted if $\sum_{i=1}^{k}(2a_i - 1) \geq 0$ for all $k = 1, 2, \cdots, 2n$. Then for each unwanted sequence σ, there exists at least one k that $\sum_{i=1}^{k}(2a_i - 1) \geq 0$. So there must be a k_0 satisfies $\sum_{i=1}^{k_0}(2a_i - 1) = -1$ and all $k = 1, 2, \cdots, k_0 - 1$ satisfies $\sum_{i=1}^{k}(2a_i - 1) \geq 0$, especially $\sum_{i=1}^{k_0}(2a_i - 1) = 0$.

Obviously k_0 must be an odd number, denoting as $k_0 = 2m+1$. So there is m 1's and m 0's in $a_1a_2\cdots a_{2m}$ and $a_{2m+1} = 0$. After reversing $a_1a_2\cdots a_{2m+1}$ of σ, we will get the sequence σ' which has $n+1$ 1's and $n-1$

0's.

On the other hand, given a sequence $\sigma' = a'_1 a'_2 \cdots a'_{2n}$ which consists of $n+1$ 1's and $n-1$ 0's. For the reason that the number of 1's is more than 0's, there must be a $k_0 = 2m+1$ that $\sum_{i=1}^{k_0}(2a_i - 1) = -1$ and $\sum_{i=1}^{k}(2a_i - 1) \geq 0$ for all $k = 1, 2, \cdots, 2m$. Reversing $a'_1 a'_2 \cdots a'_{2m+1}$ of σ', we will get a sequence σ satisfies our rules with n 1's and n 0's.

Finally, we get a one-to-one correspondence between paying patterns unwanted and all binary sequences of length $2n$ with exactly $n+1$ 1's. So the number of unwanted patterns is $\binom{2n}{n+1}$. And there is $\binom{2n}{n}$ sequences in total. Thus, $b_n = \binom{2n}{n} - \binom{2n}{n+1}$.

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