Mathematics for Computer Science
Spring 2019
Due: 23:59, April 15, 2019

Homework Set 7

Do the following special problems:

Special Problem 1 (counted as 1 exercise) In the attachment to this homework set, we give a summary of the lecture delivered today. In particular, Theorem 1 states that, under some assumptions on $f(z)$, the power series coefficients a_n of $f(z)$ can be expressed as a linear combination of residues of $f(z)/z^{n+1}$ at $z = z_j$. Assume that we have proved Theorem 1.
Question: Prove Corollary 2.

Special Problem 2 (counted as 2 exercises) Let

$$f(z) = \frac{3}{(z-5i)^2}.$$

(a) Let $n > 0$. Determine the residue of $f(z)/z^{n+1}$ at $z = 5i$.
(b) Use Theorem 1 to determine the power series expansion of $f(z) = \sum_{n\geq 0} a_n z^n$. Does this agree with the answer you would get if you apply Newton's binomial theorem to the expression $(1 - z/5i)^{-2}$?

Special Problem 3 (counted as 3 exercises) Let $A(z) = \frac{1}{\lambda - e^z}$ be a function over the complex plane, where $\lambda > 1$ is a real number.
(a) Where are all the singularities of A on the complex plane? Are they isolated singularities? Determine the residue of A at each of its isolated singularities.
(b) Consider the power series expansion $A(z) = \sum_{n\geq 0} a_n z^n$ in the neighborhood of $z = 0$. Find a closed-form expression $g(n)$ in variable n, such that $\lim_{n\to\infty} \frac{a_n}{g(n)} = 1$. You should give your reasoning rigorously.
(c) Consider the following recurrence relation: $b_0 = 1$, and for $n \geq 1$,

$$b_n = \sum_{0\leq k\leq n-1} b_k \binom{2n}{2k}.$$

Find a closed-form expression $h(n)$ in variable n, such that $\lim_{n\to\infty} \frac{b_n}{h(n)} = 1$.

Lecture Notes on Power Series Expansion for tan z

A complex function $f(z)$ has a *pole singularity* at z_0, if for some $\epsilon > 0$, $f(z)$ can be written as a convergent series

$$f(z) = \sum_{1 \le j \le m} \frac{c_j}{(z - z_0)^j} + \sum_{n \ge 0} b_n (z - z_0)^n,$$

for all z satisfying $0 < |z - z_0| < \epsilon$, where m, c_j, b_n are constants. We call m the *order* of the pole, and c_1 the *residue* of the pole at z_0. The pole is *simple*, if its order is $m = 1$.

Let $R > 0$ be any positive real number. Define the *symmetric rectangle* C_R as the set of $\{z = x + iy \mid \max\{x, y\} = R\}$. That is, C_R is the boundary of the $2R \times 2R$ square, centered at the origin in the complex plane and with its sides parallel to the x and y axes.

Theorem 1 Let $f(z)$ be a complex function with only isolated pole singularities z_i with $0 < |z_1| \le |z_2| \le \cdots$. Assume that there exists a sequence of symmetric rectangles C_{R_i}, $R_i \to \infty$ as $i \to \infty$, such that $f(z) \le \beta$ for some fixed constant $\beta > 0$ for all $z \in C_{R_i}$. Then the power series expansion of f at $z = 0$, $f(z) = \sum_{n \ge 0} a_n z^n$ satisfies for all $n > 0$

$$a_n = -\sum_{i \ge 1} (\text{Residue of } \frac{f(z)}{z^{n+1}} \text{ at } z = z_i).$$

Corollary 2 If all poles are simple, then for all $n > 0$

$$a_n = -\sum_{i \ge 1} \frac{r_i}{z_i^{n+1}},$$

where r_j is the residue of $f(z)$ at z_i.

We now apply the above corollary to analyze the power series coefficients for $\tan z$.

Fact 1 The function $\tan z$ has only simple pole singularities, located at $z_j = (j + \frac{1}{2})\pi$ for integers j and with residues r_j all equal to -1.

Proof By definition $\tan z = \sin z / \cos z = (e^{iz} - e^{-iz})/i(e^{iz} + e^{-iz})$. This implies,

$$\tan z = -i + 2i \frac{1}{1 + e^{2iz}}. \tag{1}$$

Thus, the singularities of $\tan z$ are where $1 + e^{2iz} = 0$, i.e. at z_j for all integers j. Note, with $\Delta = z - z_j$, we have in some small neighborhood $|\Delta| \le \epsilon$,

$$1 + e^{2iz} = 1 - e^{2i\Delta} = -2i\Delta(g(\Delta)),$$

where $g(0) = 1$ and $g(\Delta) = \sum_{n\geq 0} \frac{1}{(n+1)!}(2i\Delta)^n$ is non-zero and differentiable. This gives

$$\tan z = -i - \frac{1}{\Delta} + \text{ power series in } \Delta.$$

Fact 1 follows. Q.E.D.

Before invoking Corollary 2 to determine the power series for $f(z) = \tan z$, we need to show that there exists a sequence of large rectangles C_{R_i} on which $f(z)$ has values bounded by a constant β. Consider $R_j = j\pi$, and the symmetric rectangles C_{R_j}.

Fact 2 Let $j > 0$ be any integer. Then $|\tan z| < 5$ for any z on C_{R_j}.

Proof It suffices to prove that $|1 + e^{2iz}| > 1/2$ for any such z, because of (1).
Case (a) $z = j\pi + iy$, or $-j\pi + iy$: Then $|1 + e^{2iz}| = 1 + e^{-2y} > 1$.
Case (b) $z = x + j\pi$, or $x - j\pi$: Then $|1 + e^{2iz}| \geq \min\{1 - e^{-2j\pi}, e^{2j\pi} - 1\} > 1/2$. This proves Fact 2. Q.E.D.

We now use Corollary 2 to derive an expression for the power series coefficients of $\tan z = \sum_{n\geq 0} b_n z^n$. Clearly $b_0 = \tan 0 = 0$. For $n \geq 1$, we obtain

$$b_n = \sum_{\text{integers } j} \frac{1}{((j + \frac{1}{2})\pi)^{n+1}},$$

which gives $b_n = 0$ for all even integers n (For each $j \geq 0$, the terms j and $-(j + 1)$ have opposite signs and cancel out.) For all odd positive integers n, we have

$$\begin{aligned} b_n &= 2\sum_{j\geq 0} \frac{2^{n+1}}{((2j + 1)\pi)^{n+1}} \\ &= 2\left(\frac{2}{\pi}\right)^{n+1} \left(\frac{1}{1^{n+1}} + \frac{1}{3^{n+1}} + \cdots + \frac{1}{(2k + 1)^{n+1}} + \cdots\right). \end{aligned} \tag{2}$$

The above expression represents a_n nicely, even the first few terms give a fairly good approximation. In fact for large n this gives an extremely accurate asymptotic form $b_n \approx 2\left(\frac{2}{\pi}\right)^{n+1}$.