Mathematics for Computer Science: Homework 7

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Special Problem 1

In the attachment to this homework set, we give a summary of the lecture delivered today. In particular, Theorem 1 states that, under some assumptions on $f(z)$, the power series coefficients a_n of $f(z)$ can be expressed as a linear combination of residues of $f(z)/z_{n+1}$ at $z = z_j$. Assume that we have proved Theorem 1.

Question: Prove Corollary 2.

Answer:

From 5.1 of 《复变函数简明教程》, we know for $z_0 \neq \infty$ is a m order pole singularity of $f(z)$, residue of $f(z)$ at $z = z_0$ is:

$$Res(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)].$$

So with Theorem 1, power expansion of f at $z = 0$, $f(z) = \sum_{n \geq 0} a_n z^n$ having (Since the power expansion is from $z = 0$)

$$\begin{aligned}
a_n &= -\sum_{i \geq 1} Res(\frac{f(z)}{z^{n+1}}, z_i) \\
&= -\sum_{i \geq 1} \frac{1}{(m-1)!} \lim_{z \to z_i} \frac{d^{m-1}}{dz^{m-1}}[(z - z_i)^m \frac{f(z)}{z^{n+1}}] \\
&= -\sum_{i \geq 1} \frac{1}{z_i^{n+1}} \frac{1}{(m-1)!} \lim_{z \to z_i} \frac{d^{m-1}}{dz^{m-1}}[(z - z_i)^m f(z)] \\
&= -\sum_{i \geq 1} \frac{Res(f, z_i)}{z_i^{n+1}}
\end{aligned}$$

Special Problem 2

Let

$$f(z) = \frac{3}{(z - 5i)^2}.$$

(a) Let $n > 0$. Determine the residue of $f(z)/z^{n+1}$ at $z = 5i$.

(b) Use Theorem 1 to determine the power series expansion of $f(z) = \sum_{n \geq 0} a_n z^n$. Does this agree with the answer you would get if you apply Newton's binomial theorem to the expression $(1 - z/5i)^{-2}$?

Answer:

(a) Easy to know that $z_1 = 5i$ is a second order pole singularity of f. So $f(z)/z^{n+1}$ have a second order pole singularity at $z_1 = 5i$ expect $z_0 = 0$. With the formula mentioned above, we have:

$$\begin{aligned} Res(\frac{f(z)}{z^{n+1}}, z_1) &= \frac{1}{1!} \lim_{z \to 5i} \frac{d}{dz}[(z-5i)^2 \frac{f(z)}{z^{n+1}}] \\ &= 3 \lim_{z \to 5i} \frac{d}{dz}[\frac{1}{z^{n+1}}] \\ &= -\frac{3}{25} \frac{(n+1)}{(5i)^n} \end{aligned}$$

(b) With Theorem 1 and results from (a), we have:

$$a_n = -\sum_{i \geq 1} Res(\frac{f(z)}{z^{n+1}}, z_i) = -Res(\frac{f(z)}{z^{n+1}}, z_1) = -\frac{3}{25} \frac{n+1}{(5i)^n}$$

And Newton's binomial theorem with negative exponents have:

$$(1+z)^{-k} = \sum_{n \geq 0} (-1)^n \binom{k+n-1}{n} z^n$$

So there is:

$$\begin{aligned} (1 - \frac{z}{5i})^{-2} &= \sum_{n \geq 0} (-1)^n \binom{n+1}{n} \left(-\frac{z}{5i}\right)^n \\ &= \sum_{n \geq 0} \binom{n+1}{n} (5i)^{-n} z^n \end{aligned}$$

And known that:

$$f(z) = \frac{3}{(z-5i)^2} = \frac{3}{(5i)^2} (\frac{5i - z}{5i})^{-2} = -\frac{3}{25} (1 - \frac{z}{5i})^{-2}$$

So power series expansion coefficients a'_n from Newton's binomial theorem is:

$$a'_n = -\frac{3}{25} \binom{n+1}{n} (5i)^{-n} = -\frac{3}{25} \frac{n+1}{(5i)^n} = a_n$$

Special Problem 3

Let $A(z) = \frac{1}{\lambda - e^z}$ be a function over the complex plane, where $\lambda > 1$ is a real number.

(a) Where are all the singularities of A on the complex plane? Are they isolated singularities? Determine the residue of A at each of its isolated singularities.

(b) Consider the power series expansion $A(z) = \sum_{n \geq 0} a_n z^n$ in the neighborhood of $z = 0$. Find a closed-form expression $g(n)$ in variable n, such that $\lim_{n \to \infty} \frac{a_n}{g(n)} = 1$. You should give your reasoning rigorously.

(c) Consider the following recurrence relation: $b_0 = 1$, and for $n \geq 1$,

$$b_n = \sum_{0 \leq k \leq n-1} b_k \binom{2n}{2k}.$$

Find a closed-form expression $h(n)$ in variable n, such that $\lim_{n \to \infty} \frac{b_n}{h(n)} = 1$.

Answer:

Special Problem 3 continued on next page...

(a) Easy to know that when $z = \ln\lambda + 2k\pi$ for $k \in \mathcal{Z}$, $\lambda - e^z = 0$, so $z_k = \ln\lambda + 2k\pi, k \in \mathcal{Z}$ are isolated singularities of A.

$$Res(f, \ln\lambda + 2k\pi) = Res(f, \ln\lambda) = \lim_{z\to\ln\lambda}(z - \ln\lambda)\frac{1}{\lambda - e^z} = \lim_{z\to\ln\lambda}\frac{1}{-e^z} = -\frac{1}{\lambda}$$

(b) Since $A(z) = \sum_{n\geq 0} a_n z^n$ and all singularities z_k have only one order. So a_n:

$$\begin{aligned} a_n &= -\sum_{k=-\infty}^{+\infty} Res(\frac{1}{(\lambda - e^z)z^{n+1}}, z_k) \\ &= -\sum_{k=-\infty}^{+\infty} \lim_{z\to z_k}(z - z_k)\frac{1}{\lambda - e^z}\frac{1}{z^{n+1}} \\ &= \frac{1}{\lambda}\sum_{k=-\infty}^{+\infty}\frac{1}{z_k^{n+1}} \\ &= \frac{1}{\lambda}\sum_{k=-\infty}^{+\infty}\frac{1}{(\ln\lambda + 2k\pi)^{n+1}} \end{aligned}$$

And obviously, when $n \to \infty$, terms smaller than 1 with exponent $n+1$ decreases rapidly. So with $g(n) = \frac{1}{\lambda}\frac{1}{(\ln\lambda + 2m\pi)^{n+1}}$ where:

$$|\ln\lambda + 2m\pi| = \min_k\{|\ln\lambda + 2k\pi|\}$$

For example, when $\lambda = e, \ln\lambda = 1$, $m = -1$, $g(n) = (1 - 2\pi)^{-(n+1)}/e$

(c) With

$$\begin{aligned} b_n &= \sum_{k=0}^{n-1} b_k\binom{2n}{2k} \\ &= \sum_{k=0}^{n-2} b_k\binom{2n}{2k} + n(2n-1)b_{n-1} \\ &\leq \left[\frac{(2n-1)n}{6} + n(2n-1)\right] b_{n-1} \\ &= \frac{7}{6}(2n^2 - n)b_{n-1} \end{aligned}$$

And

$$\begin{aligned} b_n &= \sum_{k=0}^{n-1} b_k\binom{2n}{2k} \\ &= \sum_{k=0}^{n-2} b_k\binom{2n}{2k} + n(2n-1)b_{n-1} \\ &\geq (2n^2 - n)b_{n-1} \end{aligned}$$

So $h(n) = (2n-1)!!n!$ is a close form that has $\lim_{n\to\infty}\frac{b_n}{h(n)} = 1$

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